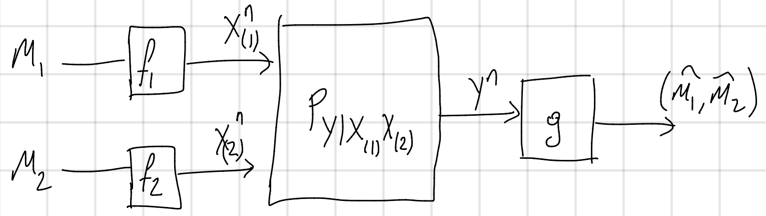
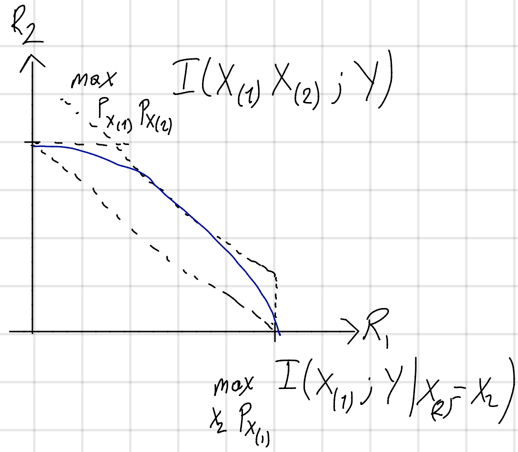
10/18/2016
Tuesday

MAC

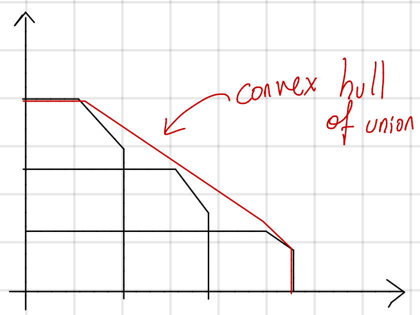




"Capacity region" (C): Closure of achievable (R_1, R_2)

Theorem: C is the convex hull of

$$C = \left\{ (R_1, R_2) : \begin{array}{l} \exists P_{X_{(1)}} P_{X_{(2)}} \text{ s.t.} \\ R_1 \leq I(X_{(1)} ; Y | X_{(2)}) \\ R_2 \leq I(X_{(2)} ; Y | X_{(1)}) \\ R_1 + R_2 \leq I(X_{(1)} X_{(2)} ; Y) \end{array} \right.$$



In fact: Instead of saying convex hull

$$C = \left\{ (R_1, R_2) : \begin{array}{l} \exists P_W P_{X_{(1)}|W} P_{X_{(2)}|W} \\ R_1 \leq I(X_{(1)} ; Y | X_{(2)}, W) \\ R_2 \leq I(X_{(2)} ; Y | X_{(1)}, W) \\ R_1 + R_2 \leq I(X_{(1)}, X_{(2)} ; Y | W) \\ |\mathcal{W}| \leq 2 \quad (\text{Alphabet of } W) \end{array} \right.$$

Proof of Achievability: Let (R_1, R_2) be in interior of region in theorem
use $P_{X_{(1)}} P_{X_{(2)}}$ from the region!

Codebooks: Generate two independent codebooks

$$\left\{ X^n_{(1)}(m_1) \right\}_{m_1=1}^{2^{nR_1}} \qquad X^n_{(1)}(m_1) \sim \prod P_{X_{(1)}} \quad \forall m_1$$

$$\left\{ X^n_{(2)}(m_2) \right\}_{m_2=1}^{2^{nR_2}} \qquad X^n_{(2)}(m_2) \sim \prod P_{X_{(2)}} \quad \forall m_2$$

Decoder: Choose unique (m_1', m_2') s.t. $(X_{(1)}^n(m_1'), X_{(2)}^n(m_2'), Y^n) \in \mathcal{T}_\varepsilon^{(n)}$

↳ we fix ε small enough.

Error: Symmetry, assume $(M_1, M_2) = (1,1)$ is transmitted.

Note $(X_{(1)}^n(1), X_{(2)}^n(1), Y^n) \sim \prod P_{X_{(1)} X_{(2)} Y}$

$X_{(1)}^n(m_1')$ and $X_{(2)}^n(m_2')$ are independent $\forall m_1' \neq 1 \quad M_2 \neq 1$

Error 1: $(X_{(1)}^n(1), X_{(2)}^n(1), Y^n) \notin \mathcal{T}_\varepsilon^{(n)}$ (Property 1)

Error 2: Not unique A) $\exists m_1' \neq 1 : (X_{(1)}^n(m_1'), X_{(2)}^{(n)}(1), Y^n) \in \mathcal{T}_\varepsilon^{(n)}$

B) $\exists m_2' \neq 1 : (X_{(1)}^n(1), X_{(2)}^n(m_2'), Y^n) \in \mathcal{T}_\varepsilon^{(n)}$

C) $\exists m_1' \neq 1, M_2 \neq 1 : (X_{(1)}^n(m_1'), X_{(2)}^n(m_2'), Y^n) \in \mathcal{T}_\varepsilon^{(n)}$

Union bound on the errors

(this was 4ε but can in fact be improved to 2ε)

$$\mathbb{P}[\text{error } 2.A)] \leq (2^{nR_1} \cancel{-1})\, 2^{-n(I(X_{(1)}; X_{(2)} Y) - 2\varepsilon)}$$

$$\leq 2^{-n(I(X_{(1)}; Y | X_{(2)}) - R_1 - 2\varepsilon)}$$

Similarly

$$\mathbb{P}[\text{error } 2.B)] \leq 2^{-n(I(X_{(2)}; Y | X_{(1)}) - R_2 - 2\varepsilon)}$$

$$\mathbb{P}[\text{error } 2.C)] \leq (2^{nR_2} \cancel{-1})(2^{nR_2} \cancel{-1})\, 2^{-n(I(X_{(1)}, X_{(2)}; Y) - 2\varepsilon)}$$

$$\leq 2^{-n(I(X_{(1)}, X_{(2)}; Y) - (R_1 + R_2) - 2\varepsilon)}$$

Last step: Existence of a codebook as good as average random.

Converse: Let ε be arb. small, assume $P_e \leq \varepsilon$

$$\begin{array}{l} M_1 \to X_{(1)}^n \searrow \\ \qquad\qquad\quad Y^n \to (\hat{M}_1, \hat{M}_2) \\ M_2 \to X_{(2)}^n \nearrow \end{array}$$

$$nR_1 = H(M_1)$$

$$= H(M_1 | X_{(2)}^n)$$

Fano handles

$$= I(M_1 ; Y^n | X_{(2)}^n) + \cancel{H(M_1 | Y^{(n)}, X_{(2)}^n)}$$

$$\leq I(X_{(1)}^n ; Y^n | X_{(2)}^n) \qquad \text{D.P.I.}$$

$$= \sum_{i=1}^n I(X_{(1)}^n ; Y_i | X_{(2)}^n, Y^{i-1})$$

$$= \sum_{i=1}^n \left(I(X_{(1)_i} ; Y_i | \underbrace{X_{(2)}^n\ Y^{i-1}}_{X_{(2)_i}}) + \cancel{I(X_{(1)}^{i-1}, X_{(1)_{i+1}}^n ; Y_i | X_{(1)_i}, X_{(2)}^n, Y^{i-1})} \xrightarrow{} 0 \right)$$

↳ Given $X_{(1)_i}$ and $X_{(2)_i}$, Y_i doesn't depend any other term inside mut info.

$$= \sum_{i=1}^n \left(I(X_{(1)_i} ; Y_i | X_{(2)_i} \right)$$

Let $W \sim \text{unif}\{1, \dots, n\}$ ⫫ everything

$$X_{(1)} \triangleq X_{(1),W} \qquad X_{(2)} \triangleq X_{(2),W} \qquad Y = Y_W$$

$$nR_1 \leq n I(X_{(1),W} ; Y_W | X_{(2),W}, W) + \text{Fano}$$

$$\leq n I(X_{(1)} ; Y | X_{(2)}, W) + \text{Fano}$$

$P_{Y|X_{(1)} X_{(2)}}$ consistent with channel!

Same argument for nR_2 ✓

$$n(R_1+R_2) = H(M_1, M_2)$$

$$= I(M_1, M_2; Y) + \underbrace{H(M_1, M_2 \mid Y)}_{\text{Fano}}$$

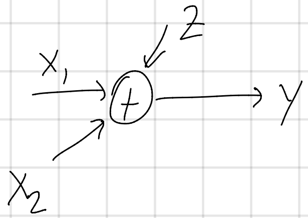
$$\leq I(X_{(1)}, X_{(2)}; Y) \quad (D.P.I)$$

Rest is like point to point version

Notice $X_{(1)} - W - X_{(2)}$

Two quick applications:

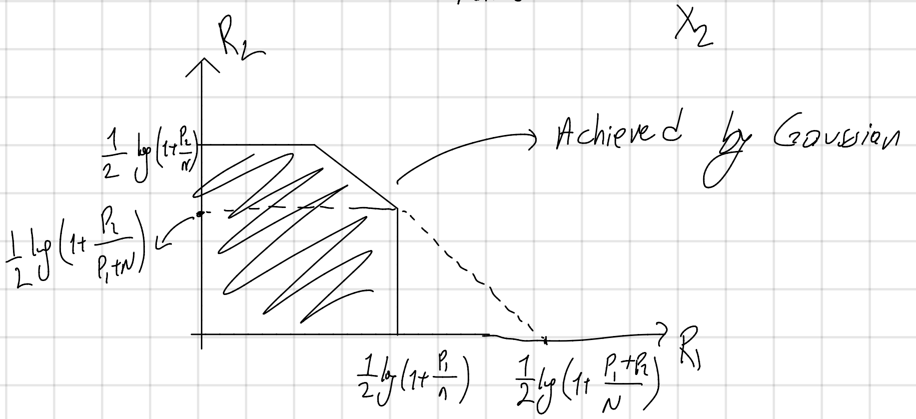
Example 1: Gaussian Channel



Power constraint

$$\frac{1}{n}\sum X_{(1),i}^2 \leq P_1$$

$$\frac{1}{n}\sum (X_{(2),i})^2 \leq P_2$$



Example 2: Binary Real addition

(No noise!) $Y = X_{(1)} + X_{(2)}$ where $X_{(1)}$ and $X_{(2)} \in \{0,1\}$, $Y \in \{0,1,2\}$

$$R_1 \leq H(Y|X_2) = H(X_1|X_2) = H(X_{(1)}) \leq 1 \text{ bit}$$

$$R_2 \leq H(X_{(2)}) \leq 1 \text{ bit}$$

$$R_1 + R_2 \leq H(X_{(1)} + X_{(2)}) \leq 1.5 \text{ bits}$$